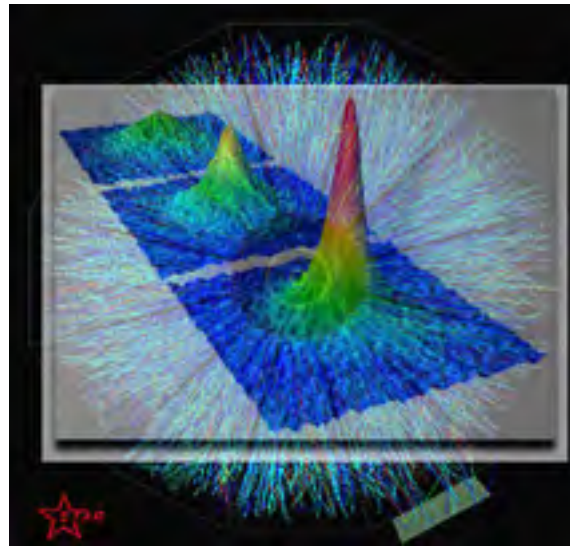


# Bose-Einstein Condensation, Isotropization, and Thermalization in Overpopulated Systems



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*Research Supported by NSF*



# OUTLINE

- Overpopulated Glasma & Bose-Einstein Condensation
- The Dynamical Onset of BEC
- Thermalization in Overpopulated Scalar System
- Isotropization in Overpopulated Scalar System
- Summary

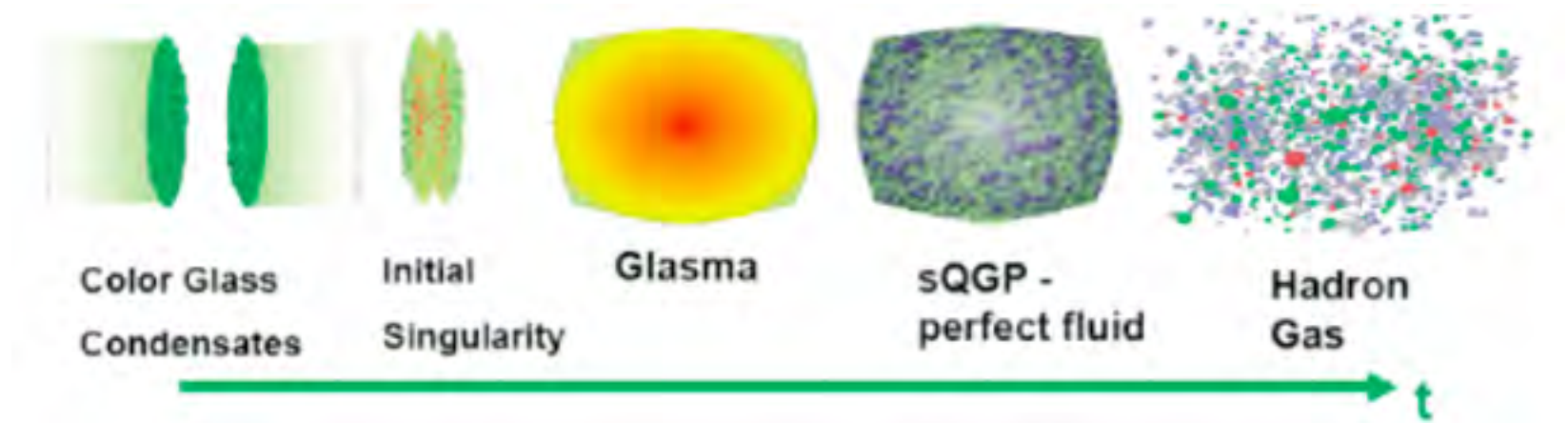
## *References:*

*Blaizot, JL, McLerran, Nucl. Phys. A920, 58(2013);*

*Huang & JL, arXiv: 1303.7214; arXiv:1402.5578[invited review for IJMPE];*

*Blaizot, Gelis, JL, McLerran, Venugopalan, Nucl. Phys. A873, 68 (2012).*

# Approach to Hydro Onset: How?



$$Q_s \sim 10\Lambda_{QCD}$$

*it should be amenable to a weakly coupled description*

$$A \sim 1/g$$

*initially dominated by strong classical field*

?



How thermalization happens?  
And quickly??

$$T_{max} \sim 2\Lambda_{QCD}$$

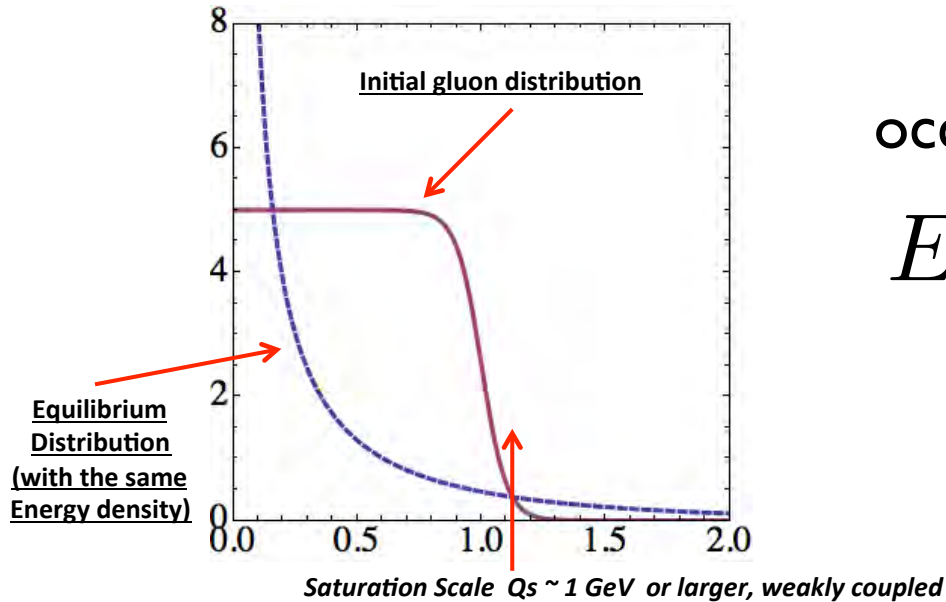
*it is plausibly a strongly coupled plasma*

$$f \sim 1$$

*dominated by quanta*

# Overpopulated Glasma

The precursor of a thermal quark-gluon plasma, known as glasma, is born as a gluon matter with **HIGH OVERPOPULATION**:



Very large  
occupation number

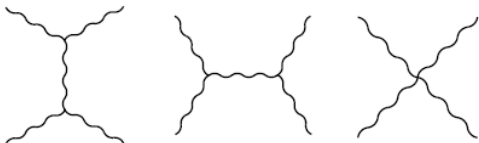
$$f \sim \frac{1}{\alpha_s}$$

$$E \sim \Lambda$$

$$M_D \sim (g f^{1/2}) \Lambda$$

$$\Lambda_s \sim (g^2 f) \Lambda$$

$$s \sim \int_p [ (1 + f) * \text{Ln} (1 + f) - f * \text{Ln} (f) ]$$



$$f * f * \alpha_s^2 \sim O(1)$$

Key observations:  
scale separation;  
 $O(1)$  scattering rate  
—> scaling solutions

# Unexpected “Detour”: BEC

We started out to derive a kinetic equation and solve it for verifying our expected thermalization via scaling solution...

$$\mathcal{D}_t f(\vec{p}) = \xi \left( \Lambda_s^2 \Lambda \right) \vec{\nabla} \cdot \left[ \vec{\nabla} f(\vec{p}) + \frac{\vec{p}}{p} \left( \frac{\alpha_S}{\Lambda_s} \right) f(\vec{p}) [1 + f(\vec{p})] \right]$$

$$\Lambda \left( \frac{\Lambda_s}{\alpha_S} \right)^2 \equiv (2\pi^2) \int \frac{d^3 p}{(2\pi)^3} f(\vec{p}) [1 + f(\vec{p})]$$

$$\Lambda \frac{\Lambda_s}{\alpha_S} \equiv (2\pi^2) 2 \int \frac{d^3 p}{(2\pi)^3} \frac{f(\vec{p})}{p}$$

Two important scales:

hard scale **Lambda**

soft scale **Lambda\_s**

The numerical evolution kept blowing up despite months' struggle of finding any potential error ...

At some point we finally realized:

**THE OVERPOPULATED SYSTEM IS DRIVEN TO  
A TRUE PHYSICAL SINGULARITY**

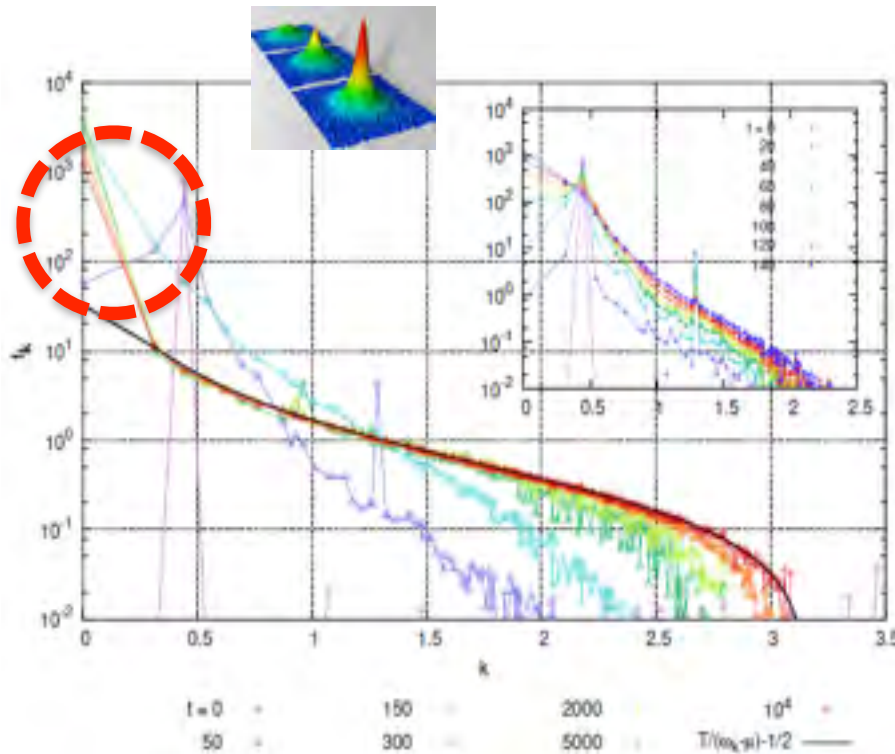
**WHERE BEC OCCURS!**

# STRONG EVIDENCE OF BEC FROM SCALAR FIELD THEORY SIMULATIONS

Bose–Einstein condensation and thermalization of the quark–gluon plasma

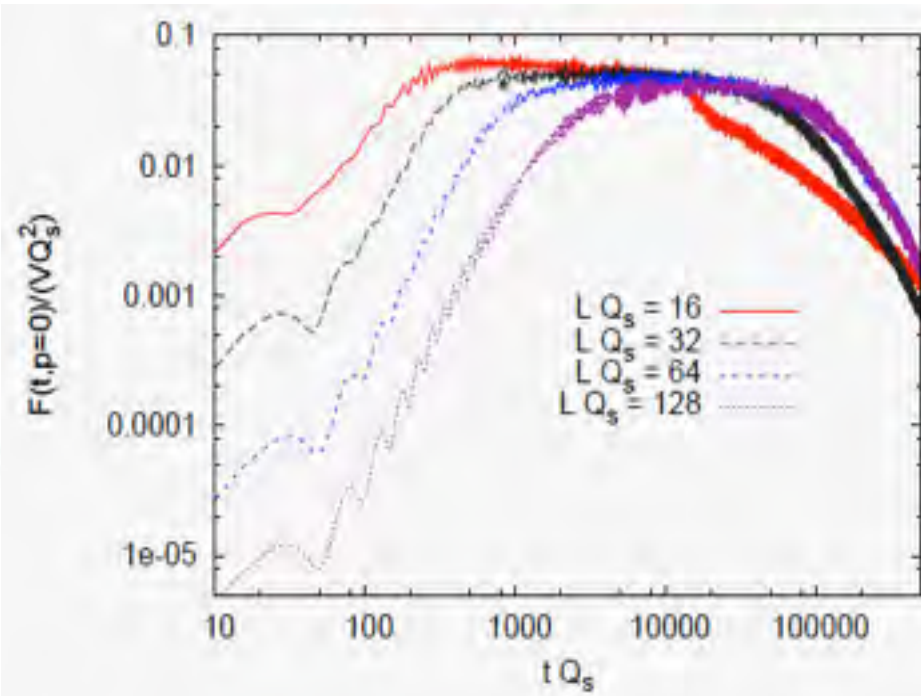
Jean-Paul Blaizot<sup>a</sup>, François Gelis<sup>a</sup>, Jinfeng Liao<sup>b,\*</sup>, Larry McLerran<sup>b,c</sup>,  
Raju Venugopalan<sup>b</sup>

*Absolutely true for pure elastic scatterings;  
True, in transient sense, for systems with inelastic processes*



*From: Epelbaum & Gelis 1107.0668*

*From: Berges & Sexty 1201.0687*



# Overpopulation: Thermodynamic Consideration

Our initial gluon system is highly **OVERPOPULATED**:

$$f(p) = f_0 \theta(1 - p/Q_s),$$
$$\epsilon_0 = f_0 \frac{Q_s^4}{8\pi^2}, \quad n_0 = f_0 \frac{Q_s^3}{6\pi^2}, \quad n_0 \epsilon_0^{-3/4} = f_0^{1/4} \frac{2^{5/4}}{3\pi^{1/2}},$$

This is to be compared with the thermal BE case:

$$n \epsilon^{-3/4}|_{SB} = \frac{30^{3/4} \zeta(3)}{\pi^{7/2}} \approx 0.28$$

Overpopulation occurs when:  $f_0 > f_0^c \approx 0.154$

Identifying  $f_0 \rightarrow 1/\alpha_s$ , even for  $\alpha_s = 0.3$ ,  
the system is highly overpopulated!!

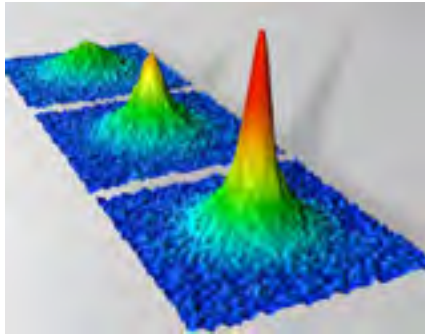
**Overpopulation  $\rightarrow$  BEC**

# BEC: Quantum Coherence $\Leftrightarrow$ Overpopulation



$$f_{\text{eq}}(\mathbf{k}) = n_c \delta(\mathbf{k}) + \frac{1}{e^{\beta(\omega_{\mathbf{k}} - m_0)} - 1}$$

*Ich behaupte, dass in diesem Falle eine mit der Gesamtdichte stets wachsende Zahl von Molekülen in den 1. Quantenzustand (Zustand ohne kinetische Energie) übergeht, während die übrigen Moleküle sich gemäss dem Parameter-Wert  $\lambda = 1$  verteilen. Die Behauptung geht also dahin, dass etwas Ähnliches eintritt wie beim isothermen Komprimieren eines Dampfes über das Sättigungsvolumen. Es tritt eine Scheidung ein; ein Teil "kondensiert", der Rest bleibt ein gesättigtes ideales Gas." ( $\lambda = 0$   $\lambda = 1$ ).*



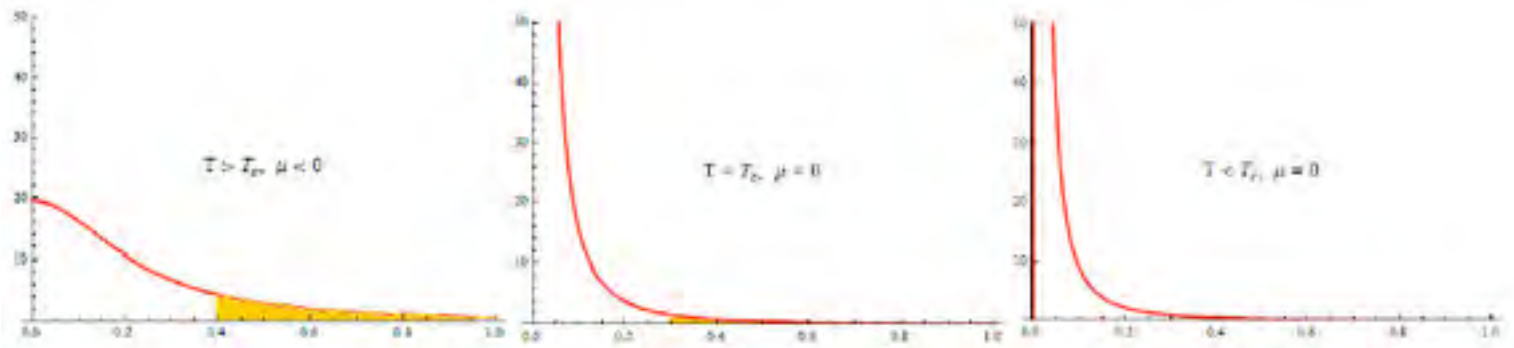
Einstein: new phase emerges with condensate, when quantum wave scale overlaps with inter-particle scale (--- the 1st application of de Broglie wavelength idea)

**Quantum Coherence** implies **OVERPOPULATION**:

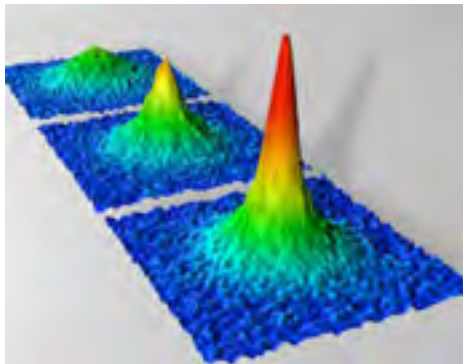
$$\frac{\lambda_{dB}}{d} \sim \left( n \epsilon^{-3/4} \right)^\alpha \sim \hat{O}(1)$$

# BEC in The Very Cold

Brilliant evaporative cooling: precisely to achieve  
**OVERPOPULATION**



*Cooling procedure: kick out fast atoms (truncating UV tail);  
then let system relax toward new equilibrium;  
relaxation via IR particle cascade & UV energy cascade.*



It took ~70 years to achieve  
**OVERPOPULATION**,  
thus BEC in *ultra-cold* bose gases.

$$n \cdot \epsilon^{-3/4} > \hat{O}(1) \text{ threshold}$$

# BEC in the Very Hot!

Temperature

$10^{-8} K$     $10^0 K$     $10^1 K$     $10^2 K$     $\sim\sim$     $10^{12} K$

---

cold  
atomic  
gas

liquid  
helium;  
  
cosmic  
axion?

magnon

cavity  
photon;  
  
magnon

overpopulated  
glasma!

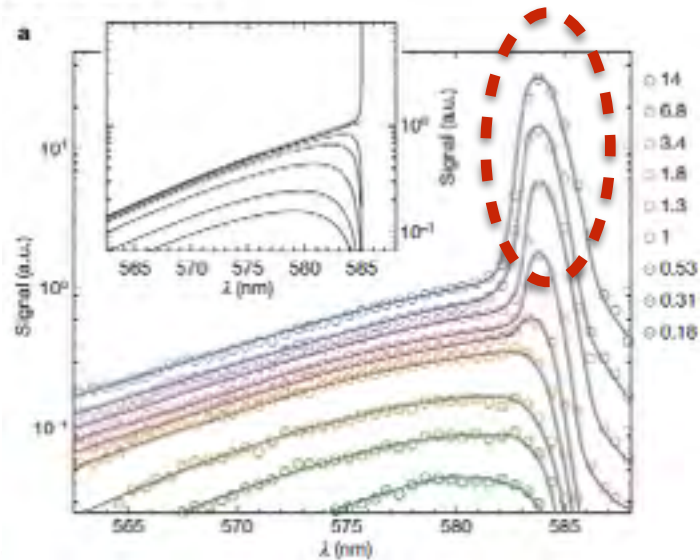
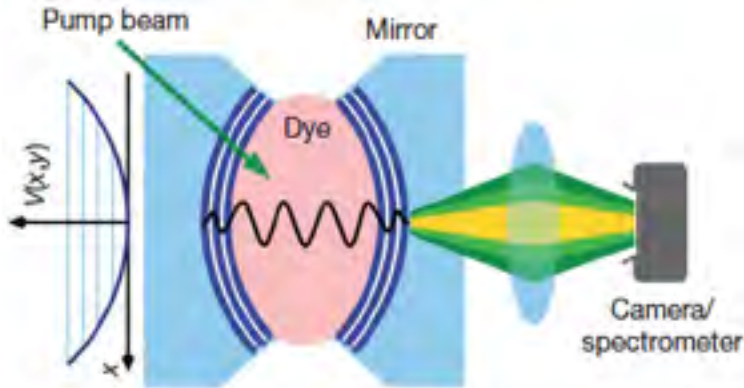
# BEC for Non-Conserved Particles

## LETTER

doi:10.1038/nature09567

### Bose–Einstein condensation of photons in an optical microcavity

Jan Klaers, Julian Schmitt, Frank Vewinger & Martin Weitz



increasing the photon density, we observe the following BEC signatures: the photon energies have a Bose–Einstein distribution with a massively populated ground-state mode on top of a broad thermal wing; the phase transition occurs at the expected photon density and exhibits the predicted dependence on cavity geometry; and the ground-state mode emerges even for a spatially displaced pump spot.

*Another example:  
idea of overcooled pion gas  
in heavy ion collisions.*

**Key point: under suitable conditions, non-conserved particles may become effectively or transiently conserved.**

# Can Kinetic Theory Describe BEC?

PHYSICAL REVIEW  
LETTERS

VOLUME 81, NUMBER 24

PHYSICAL REVIEW LETTERS

14 DECEMBER 1998

**Quantum Kinetic Theory of Condensate Growth: Comparison of Experiment and Theory**

17 APRIL 1995

PHYSICAL REVIEW B

VOLUME 15, NUMBER 1

1 JANUARY 1977

**Time evolution of a Bose system passing through the critical point**

E. Levich

ovot, Israel

Kinetics of Bose Conden

**Kinetics of the Bose-Einstein condensation**

PHYSICAL REVIEW LETTERS

PHYSICAL REVIEW LETTERS

**Initial Stages of Bose-Einstein Condensation**

**Kinetics of Bose-Einstein Condensation in a Trap**

H. T. C. Stoof

C. W. Gardiner,<sup>1</sup> P. Zoller,<sup>2</sup> R. J. Ballaoh,<sup>3</sup> and M. J. Davis<sup>3</sup>

*Institute for Theoretical Physics, University of Utrecht, Princetonplein 5,*

PHYSICAL REVIEW A **66**, 013603 (2002)

*6, 3508 TA Utrecht, The Netherlands*

*received 29 August 1996)*

**Scenario of strongly nonequilibrated Bose-Einstein condensation**

Natalia G. Berloff<sup>1,\*</sup> and Bo

PHYSICAL REVIEW B **66**, 085304 (2002)

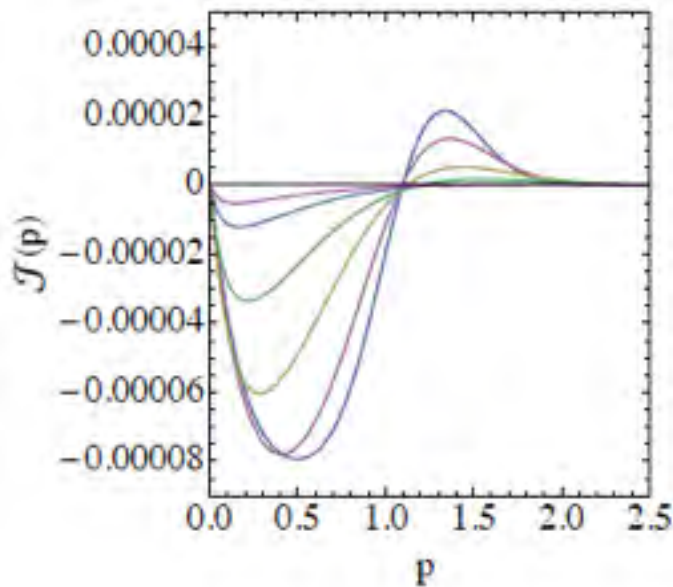
**Polariton dynamics and Bose-Einstein condensation in semiconductor microcavities**

D. Porras,<sup>1</sup> C. Ciuti,<sup>2</sup> J. J. Baumberg,<sup>3</sup> and C. Tejedor<sup>1</sup>

Kinetic description is widely used for BEC phenomena  
(trapped atoms, hard sphere gas, polaritons, cosmic scalars, ...)

# Kinetic Equations with Small Angle Scatterings

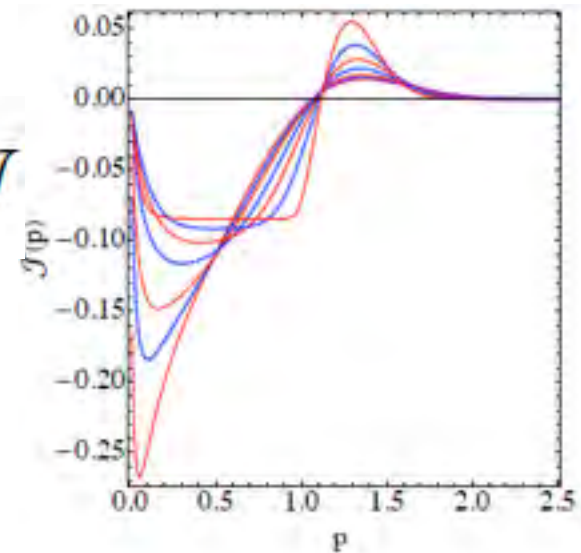
$$\mathcal{D}_t f(\vec{p}) = \xi \left( \Lambda_s^2 \Lambda \right) \vec{\nabla} \cdot \left[ \vec{\nabla} f(\vec{p}) + \frac{\vec{p}}{p} \left( \frac{\alpha_S}{\Lambda_s} \right) f(\vec{p}) [1 + f(\vec{p})] \right]$$



$f_0=0.1$  (underpopulated)

$$\mathcal{D}_t f = -\nabla \cdot \mathcal{J}$$

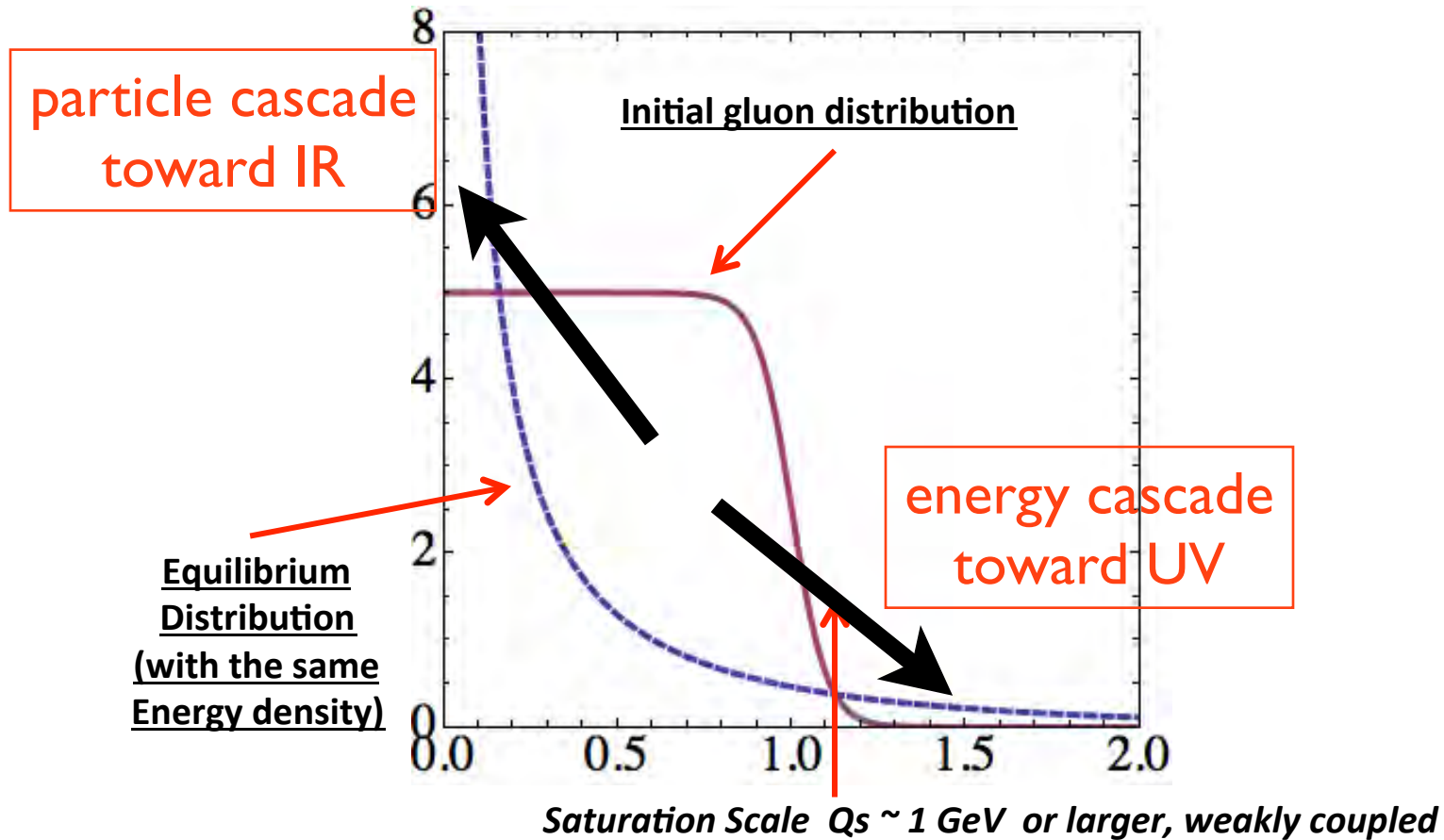
UV & IR  
cascade



$f_0=1$  (overpopulated)

*Blaizot, JL, McLerran, 1305.2119, NPA2013*

# How Thermalization Proceeds



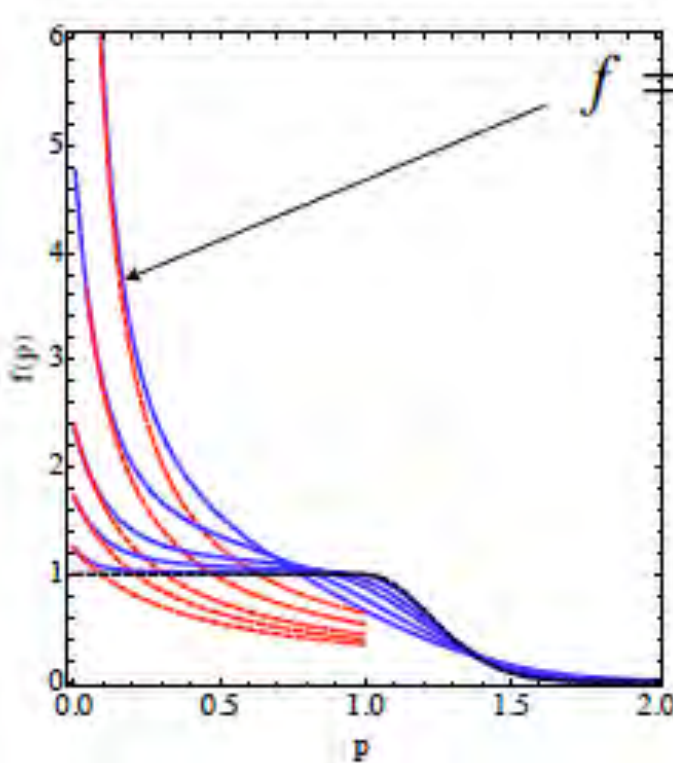
Initial glasma:  $\Lambda \sim \Lambda_s \sim Q_s$   $\longrightarrow$  Thermalized weakly-coupled QGP:  $\Lambda \sim T$   
 $\Lambda_s \sim \alpha_s * T$

separation of two scales toward thermalization

$$\frac{\Lambda_s}{\Lambda} \sim \alpha_s$$

# How BEC Onset Occurs Dynamically?

*A crucial step: rapid IR local thermalization*



$$f = \frac{T^*}{p - \mu^*} \quad (\mu^* < 0)$$

Very strong particle flux  
toward IR,  
leading to rapid growth  
and almost instantaneous  
local thermal distribution  
of very soft modes

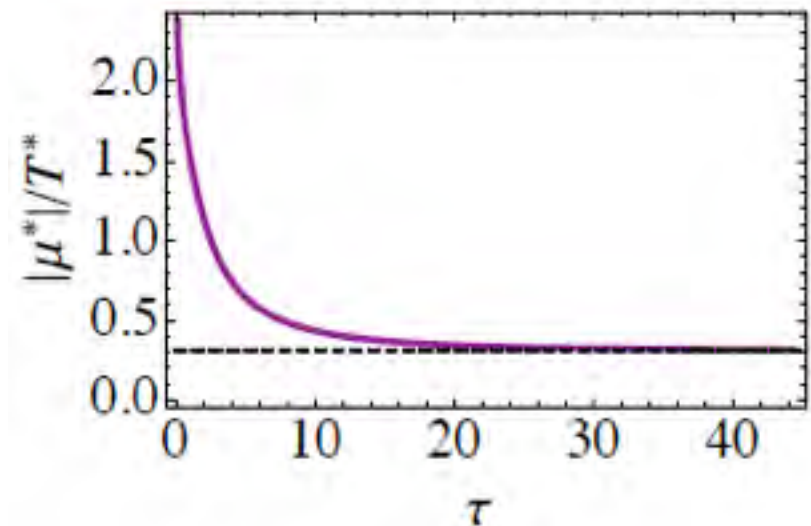
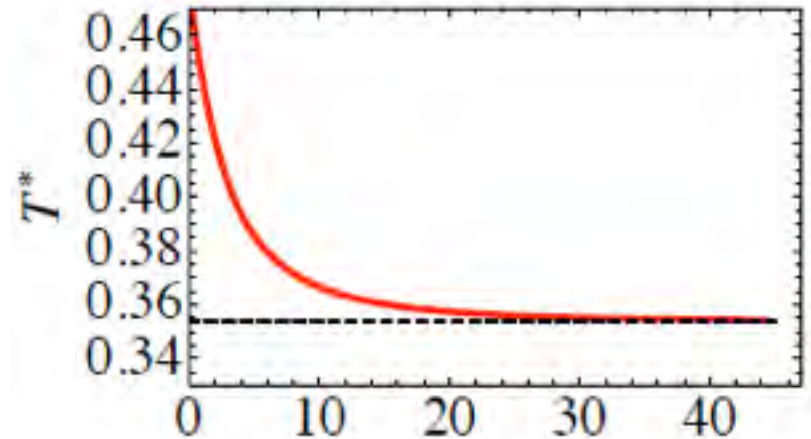
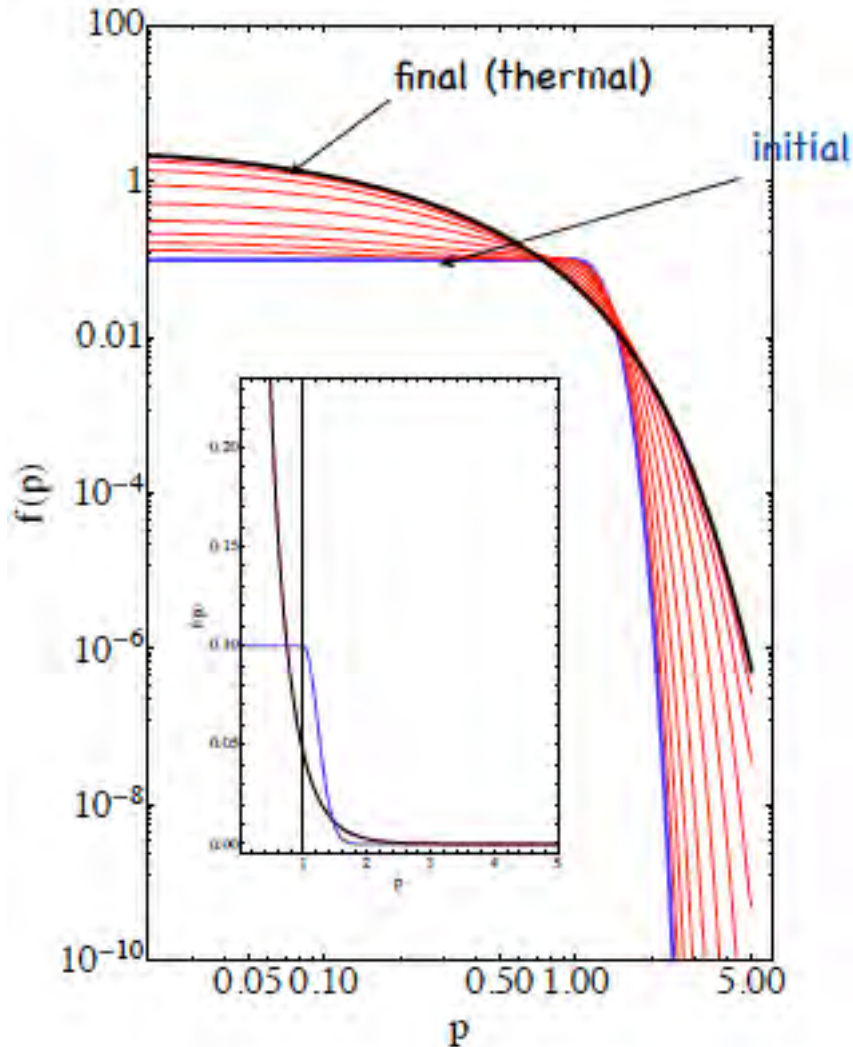
What happens next depends on **INITIAL CONDITION:**  
*underpopulation v.s. overpopulation*

*Blaizot, JL, McLerran, 1305.2119, NPA2013*

# Underpopulated Case

$$f_0 = 0.1$$

*In underpopulated case, the system thermalizes to thermal BE distribution.*

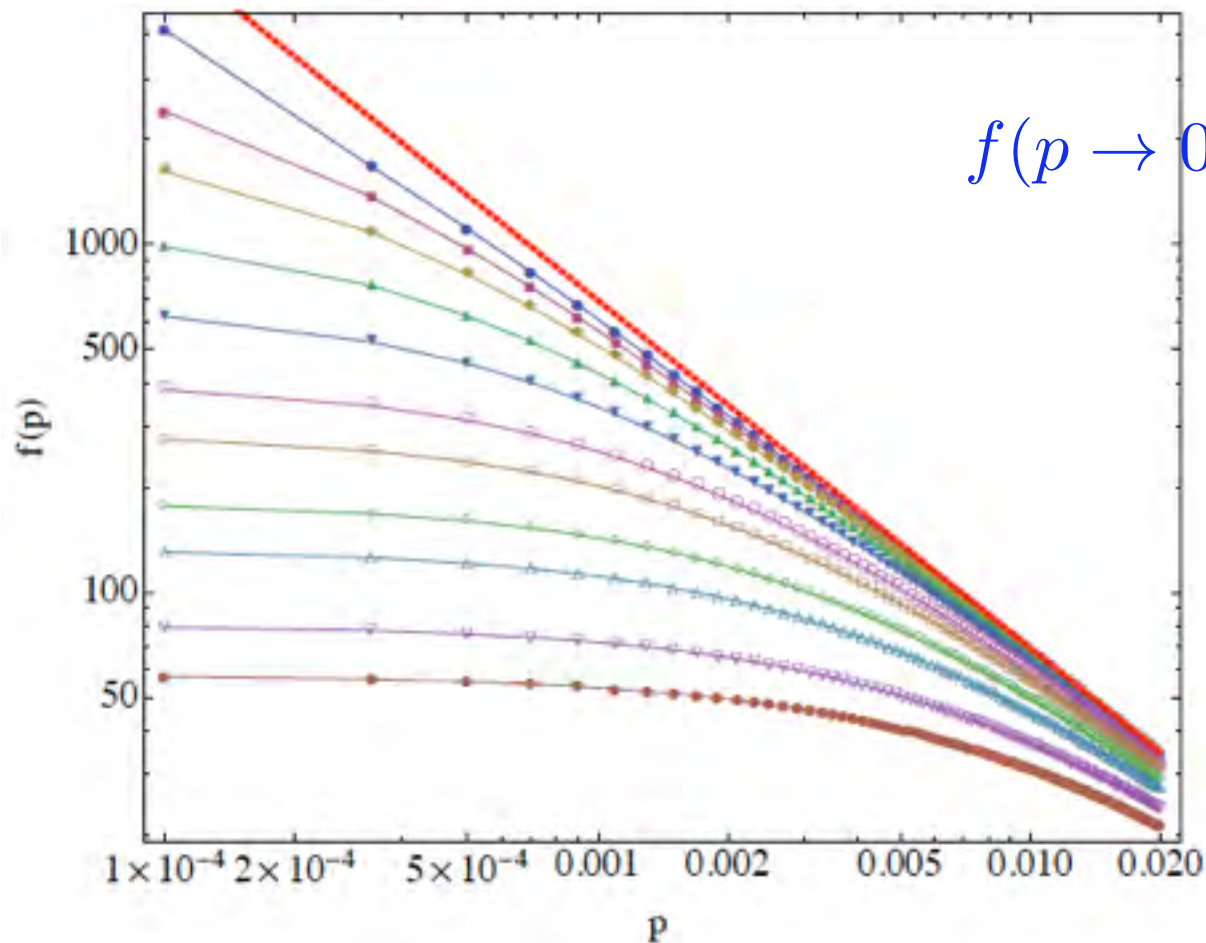


# Overpopulated Case: How Onset of BEC Develops?

Before it could reach equilibrium, onset of BEC occurs!

A **critical** IR distribution develops, i.e.  $\mu^*$  vanishes.

*(In thermal BEC: global distribution must be critical.)*

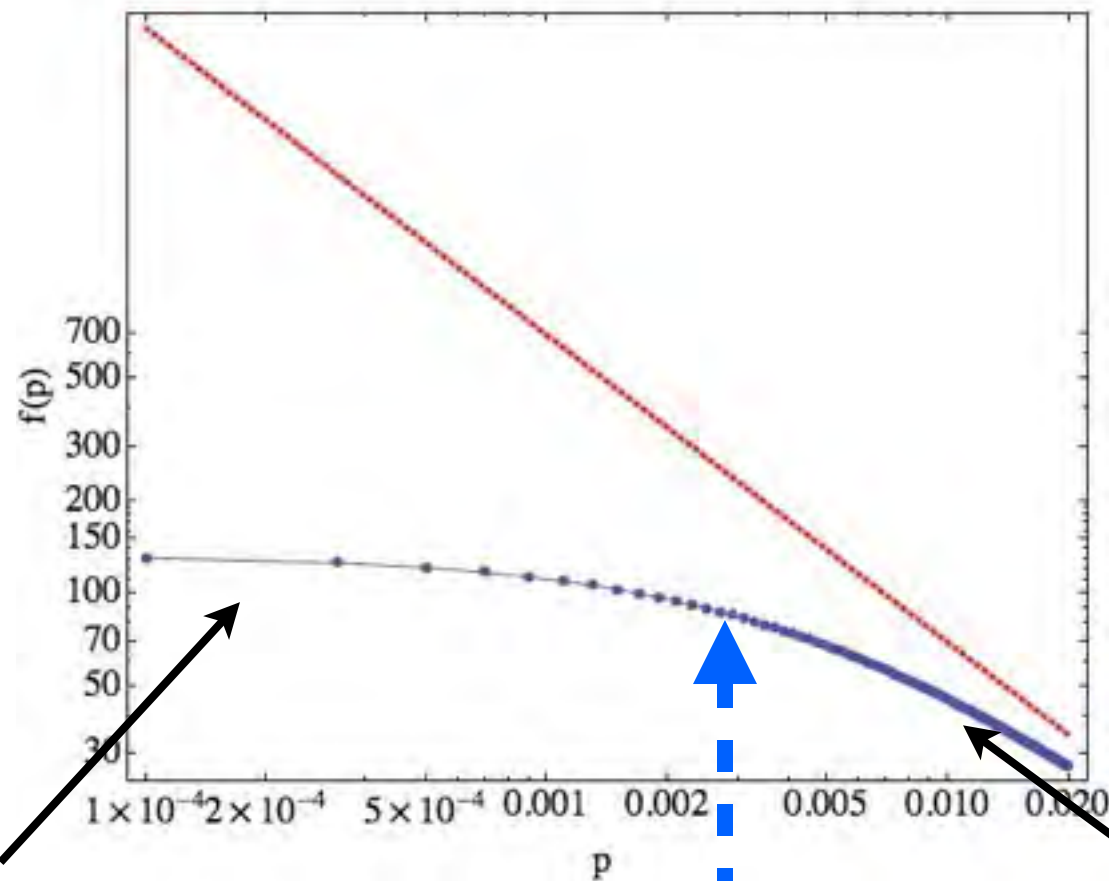


$$f(p \rightarrow 0) \rightarrow \frac{T^*}{p - \mu^*}$$

$$\mu^* \rightarrow 0$$

$$f_0 = 1$$

# Overpopulated Case: How Onset of BEC Develops?



$$p \ll \mu^*$$

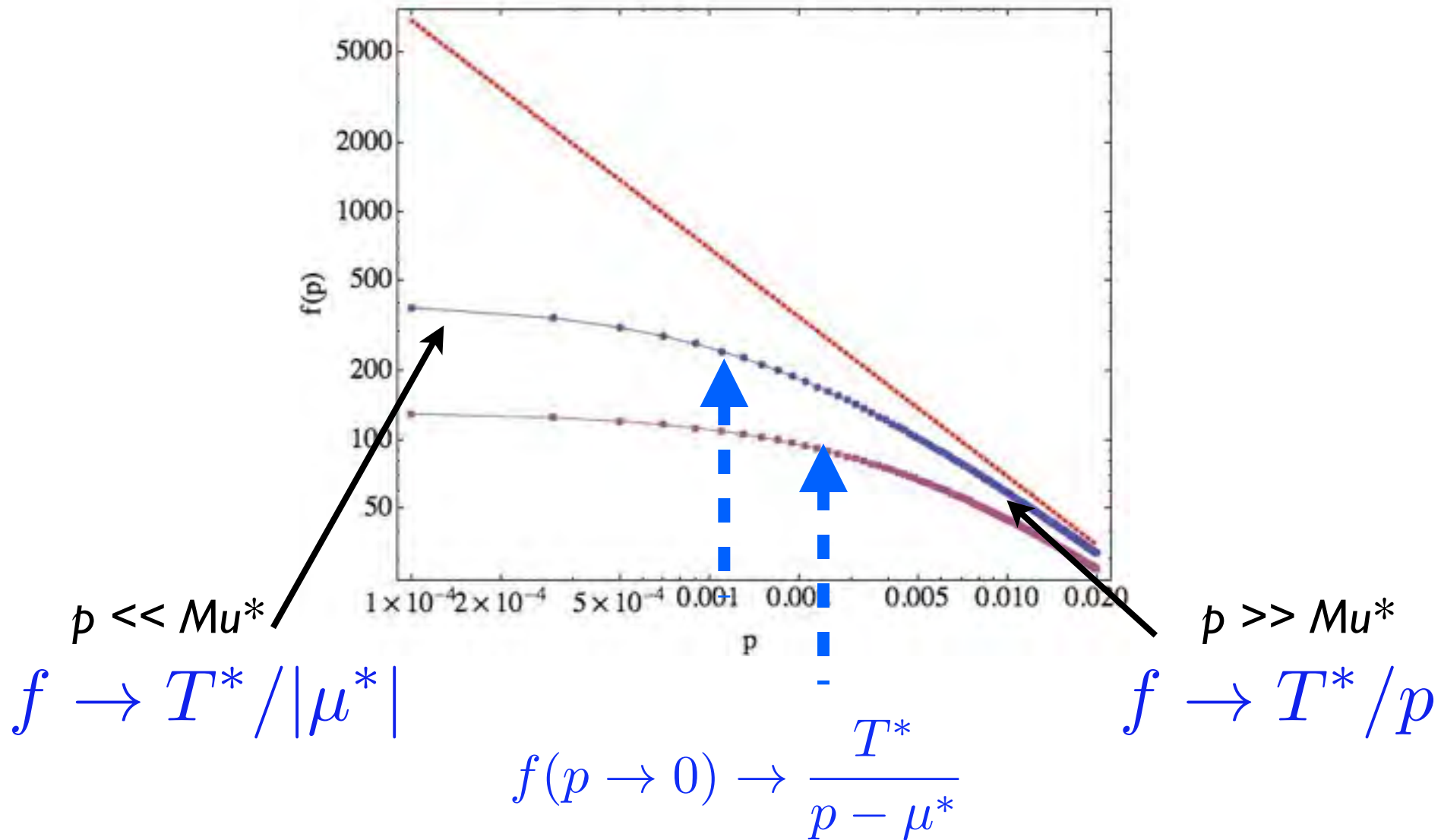
$$f \rightarrow T^* / |\mu^*|$$

$$f(p \rightarrow 0) \rightarrow \frac{T^*}{p - \mu^*}$$

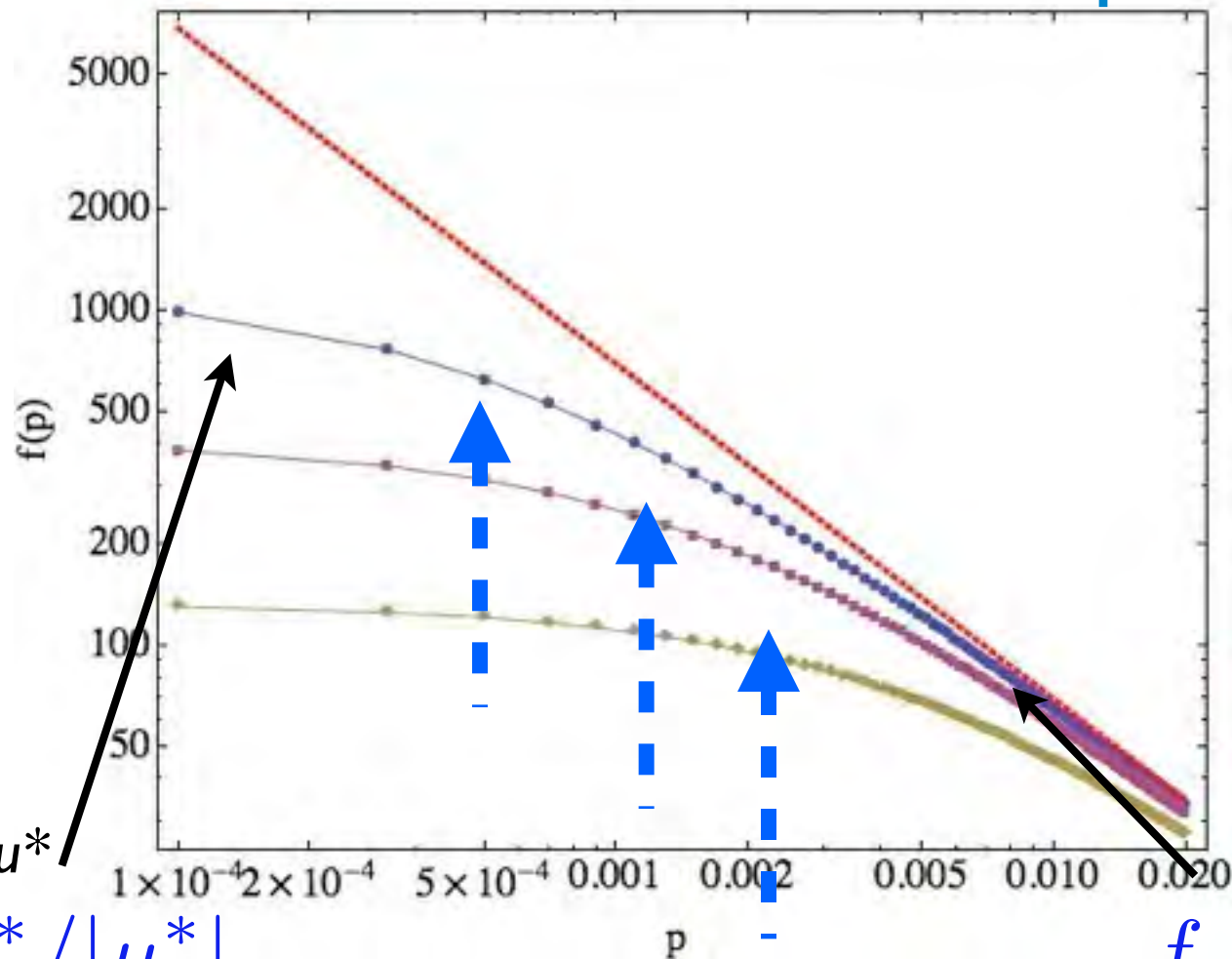
$$p \gg \mu^*$$

$$f \rightarrow T^* / p$$

# Overpopulated Case: How Onset of BEC Develops?



# Overpopulated Case: How Onset of BEC Develops?



$p \ll \mu^*$

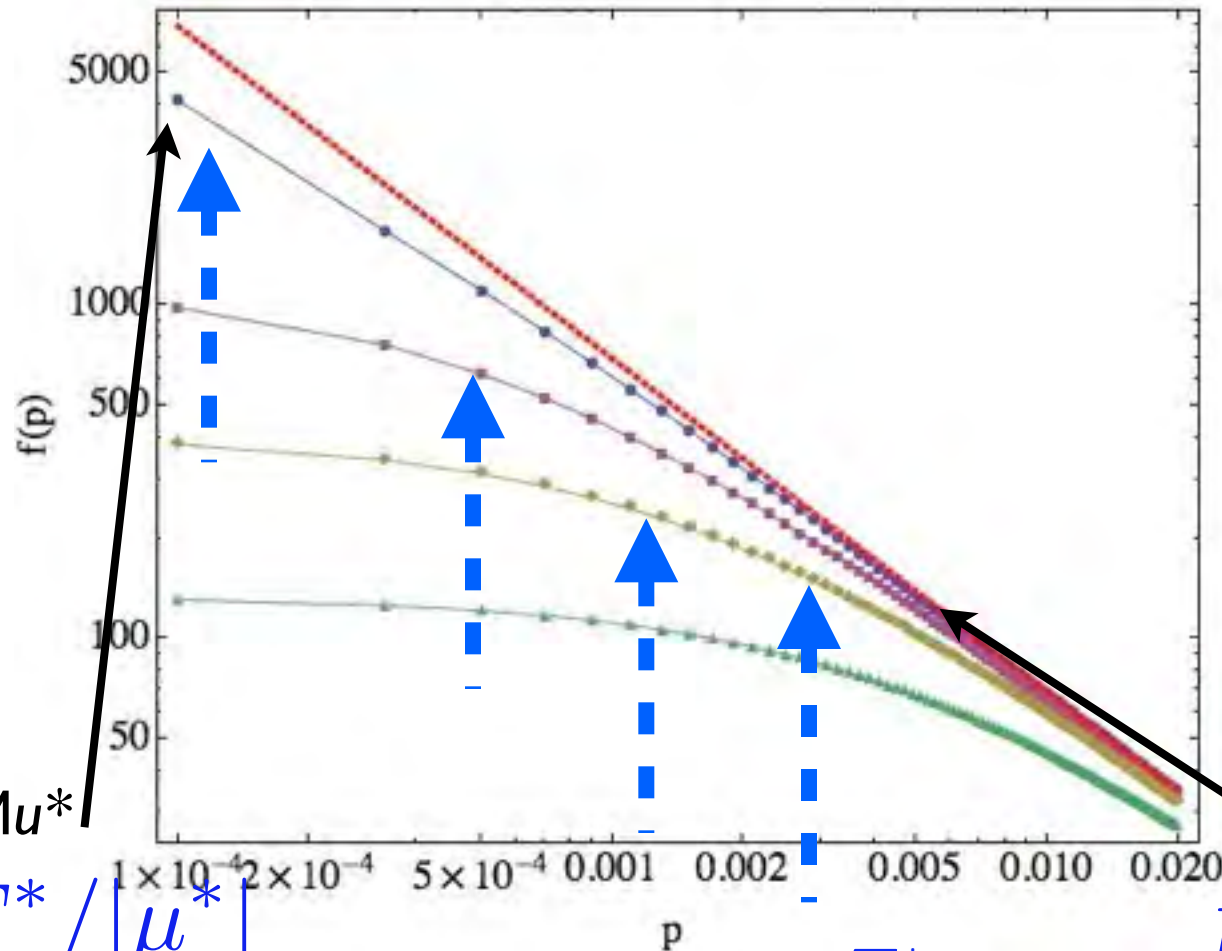
$$f \rightarrow T^* / |\mu^*|$$

$$f(p \rightarrow 0) \rightarrow \frac{T^*}{p - \mu^*}$$

$p \gg \mu^*$

$$f \rightarrow T^* / p$$

# Overpopulated Case: How Onset of BEC Develops?



$\mu^* \rightarrow 0$   
proceed in a  
self-similar  
scaling way

$p \ll M\mu^*$

$f \rightarrow T^* / |\mu^*|$

$$f(p \rightarrow 0) \rightarrow \frac{T^*}{p - \mu^*}$$

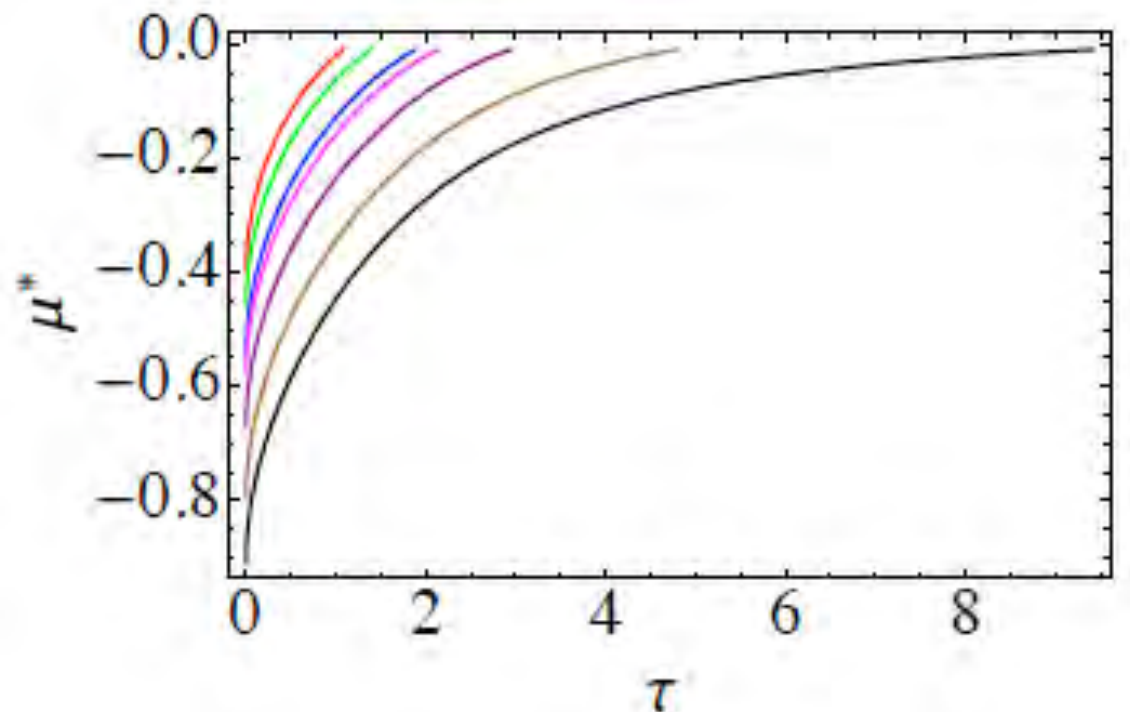
$p \gg M\mu^*$

$f \rightarrow T^* / p$

# Onset of Dynamical BEC

Onset of dynamical (out-of-equilibrium) BEC:

- \* occurring in a finite time
- \* local  $\mu^*$  vanishes with a **scaling behavior**
- \* persistence of particle flux toward zero momentum



$$|\mu^*| = C(\tau_c - \tau)^\eta.$$

$$\eta \simeq 1$$

For different  
 $f_0 = 0.2, 0.3, 0.5, 0.8, 1, 2, 5$

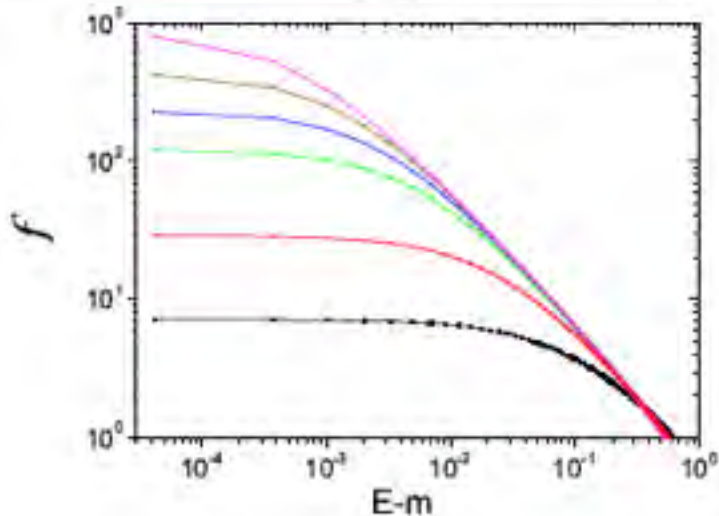
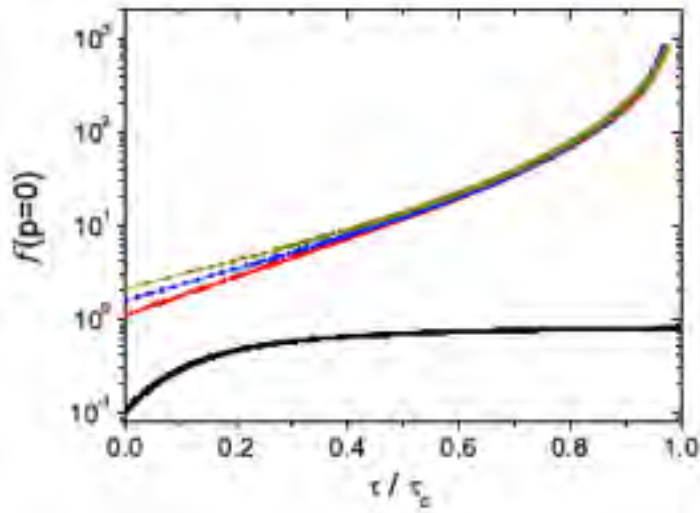
*Blaizot, JL, McLerran, I305.2119, NPA2013*

# Effects of Finite Masses

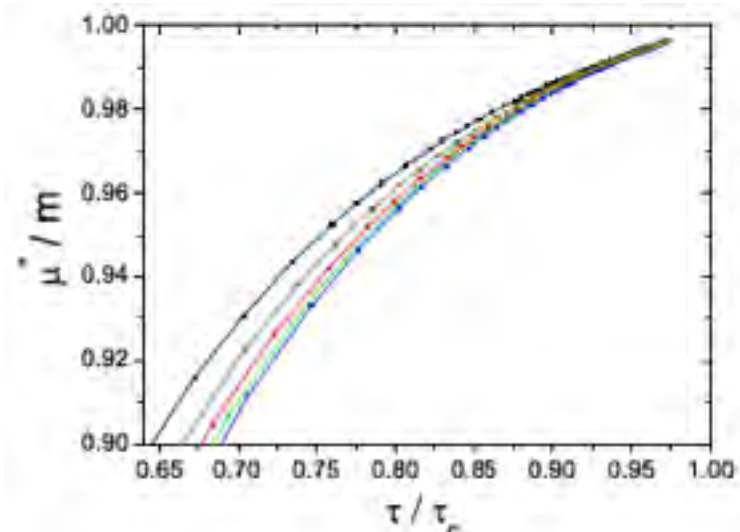
Interesting issues when there is finite external mass:

\* Onset changes,  $\mu^* \rightarrow \text{Mass}$

\* Deep IR dispersion changes,  $\sim p^2$  (NR) instead of  $\sim p$  (UR)



Interesting issues when there is finite screening mass: no more enhancement of small angle scatterings.



$$\mu = m - \lambda (\tau_c - \tau)^\eta \quad \eta \simeq 1$$

*Very similar onset dynamics as in the massless case!*

# Including the Inelastic

An inelastic kernel including  $2 \leftrightarrow 3$  processes  
(Gunion-Bertsch, under collinear and small angle approximation)

$$\mathcal{D}_t f_p = \mathcal{C}_{2 \leftrightarrow 2}^{\text{eff}}[f_p] + \mathcal{C}_{1 \leftrightarrow 2}^{\text{eff}}[f_p],$$

*Huang & JL, arXiv:1303.7214*

$$\begin{aligned} \mathcal{C}_{1 \leftrightarrow 2}^{\text{eff}} = & \xi \alpha_s^2 R \frac{I_a}{I_b} \left\{ \int_0^{z_c} \frac{dz}{z} [g_p f_{(1-z)p} f_{zp} - f_p g_{(1-z)p} g_{zp}] \right. \\ & \left. + \int_0^{z_c} \frac{dz}{(1-z)^4 z} [g_p g_{zp/(1-z)} f_{p/(1-z)} - f_p f_{zp/(1-z)} g_{p/(1-z)}] \right\} \end{aligned}$$

A number of features:

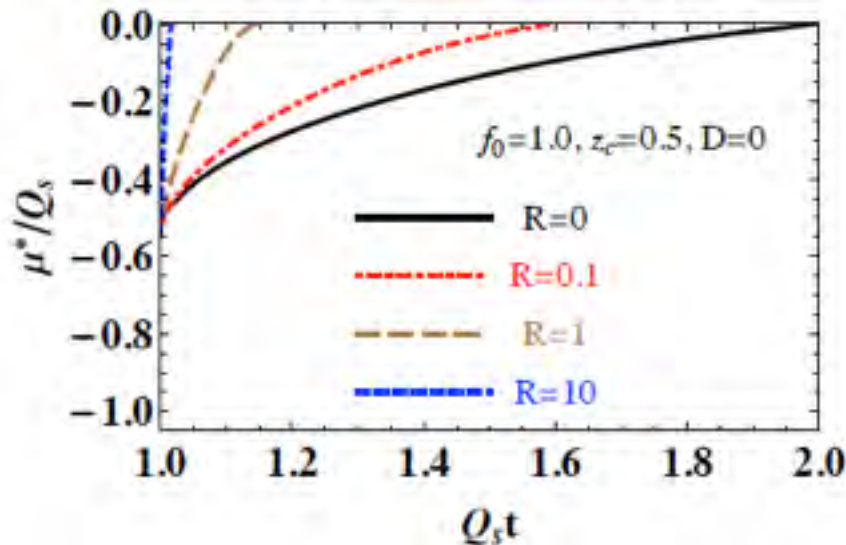
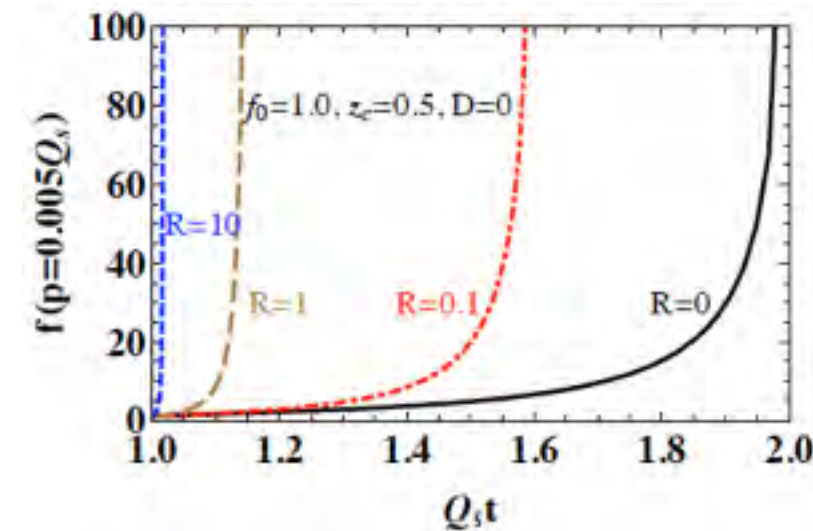
- \* fixed point: BE distribution with zero chemical potential
- \* always positive at very small momentum
- \* purely inelastic case --- correctly thermalize to BE

The question changes now:

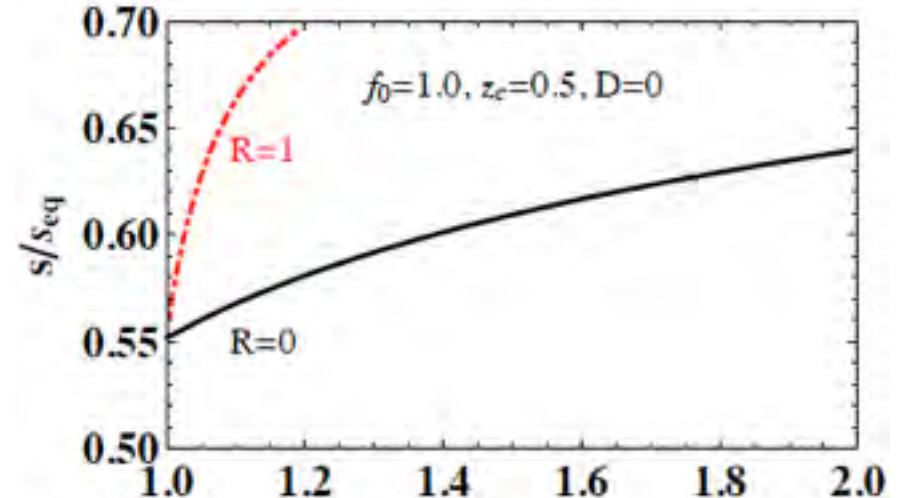
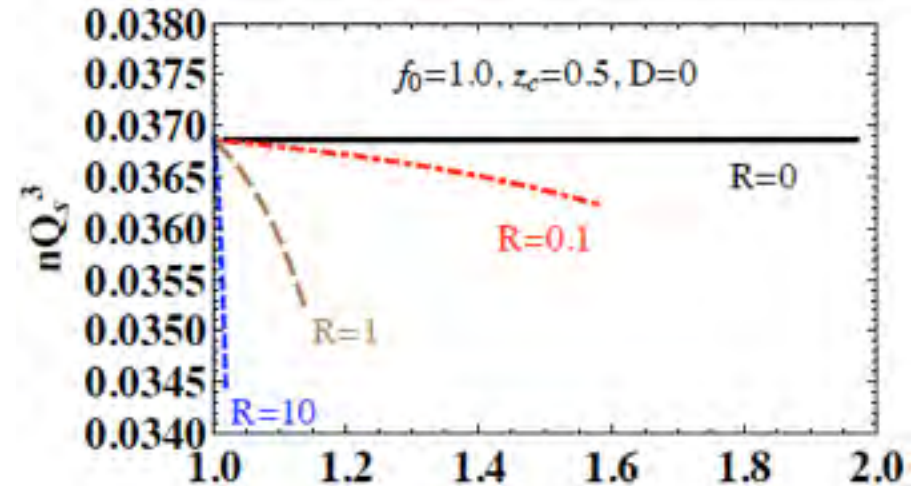
no condensate in thermal states,  
but dynamical BEC while still far from being thermal.

# Effects from the Inelastic

Local effect: enhance IR growth, accelerate the onset



Global effect: reduce number density, enhance entropy growth

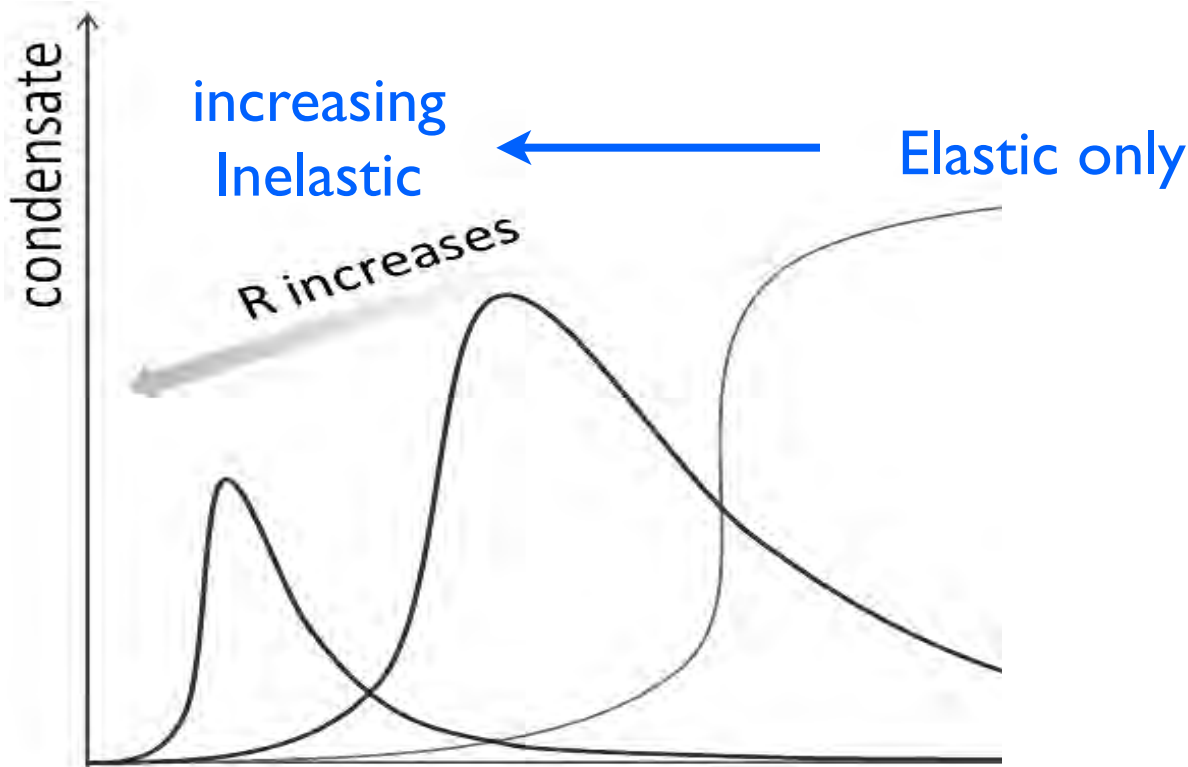


$R$ : ratio of the inelastic to the elastic kernel

Huang & JL, arXiv:1303.7214

# The “Fuller” Picture

What we find: the inelastic process catalyzes  
the onset of dynamical (out-of-equilibrium) BEC.  
It might sound contradicting with common wisdom ...  
but it is NOT.

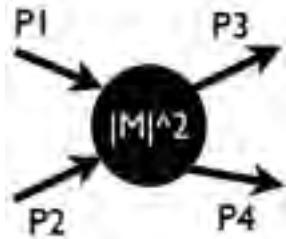


# Evolution beyond Onset

- \* To evolve the system beyond onset, one needs a set of kinetic equations describing the co-evolution of condensate + gluons.
- \* It is difficult (at the moment) to do that for the gauge field system.
- \* We instead study the SCALAR SYSTEM to explore the interesting interplay between condensate and particles toward thermalization.

Kinetic equations for scalar system:  $|\mathcal{M}|^2 = \lambda^2 \leftarrow \text{this matters!}$

$$f(\vec{p}) = n_c (2\pi)^3 \delta^{(3)}(\vec{p}) + g(\vec{p}) \quad (2\pi)^3 \delta^{(3)}(\vec{p}_1) \mathcal{D}_t n_c + \mathcal{D}_t g(\vec{p}_1) = C[n_c, g(\vec{p})]$$



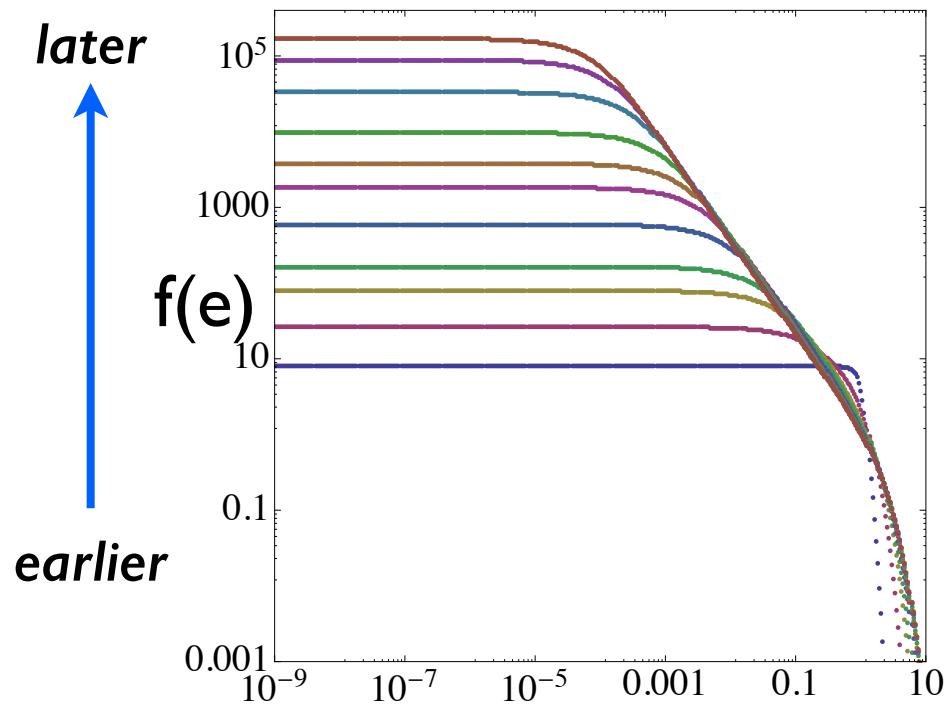
$$\mathcal{D}_t g(\vec{p}_1) = C_0[g(\vec{p})] + C_2[n_c, g(\vec{p})] + C_3[n_c, g(\vec{p})] + C_4[n_c, g(\vec{p})]$$

$$\mathcal{D}_t n_c = \tilde{C}_1[g(\vec{p})]$$

Two types of fixed points from under-/over-populated initial conditions:

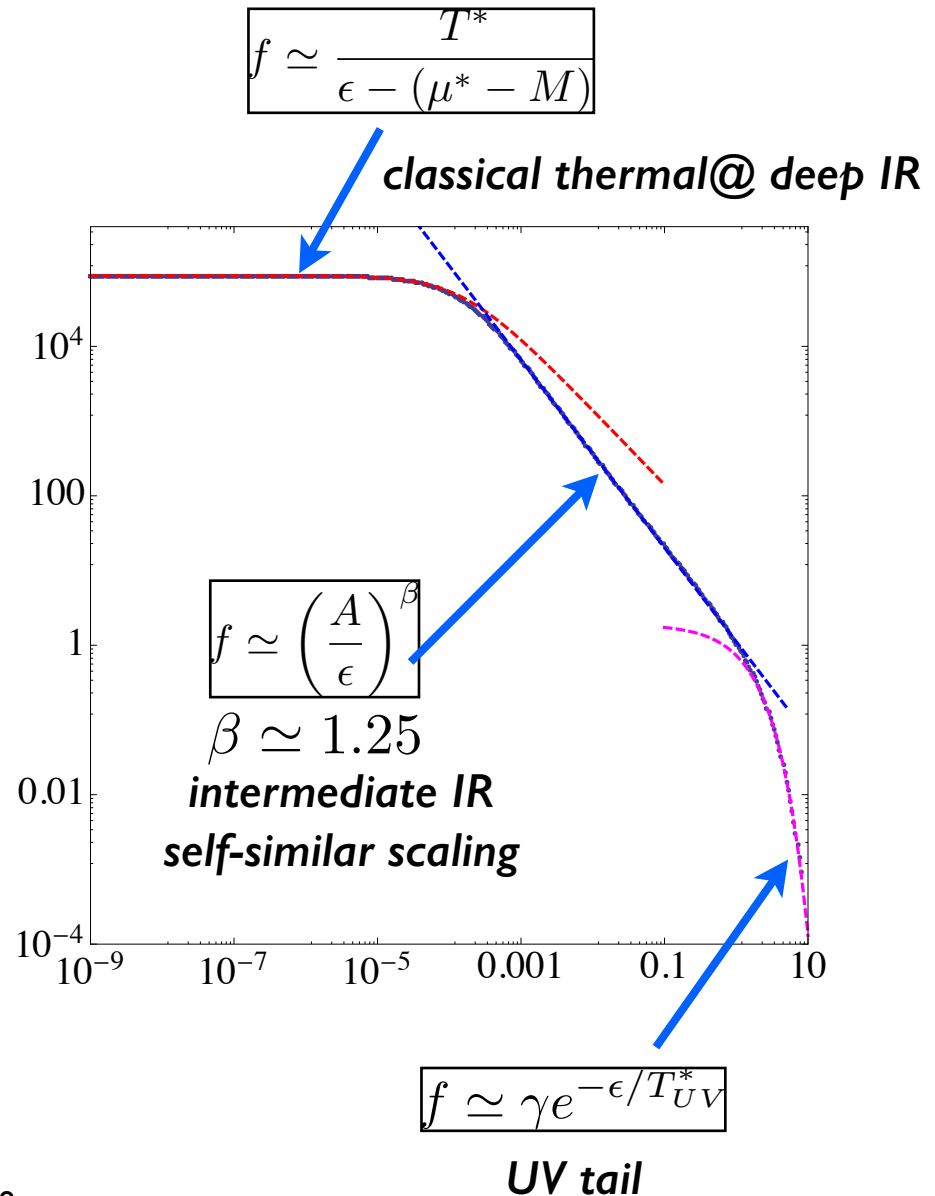
- (1) a Bose-Einstein distribution  $g_{BE} = \frac{1}{e^{(E-\mu)/T}-1}$  with any  $\mu \leq M$  and zero condensate  $n_c = 0$ ;
- (2) a Bose-Einstein distribution  $g_{BE} = \frac{1}{e^{(E-\mu)/T}-1}$  with  $\mu = M$  and a nonzero condensate  $n_c > 0$ .

# Evolution before Onset of BEC

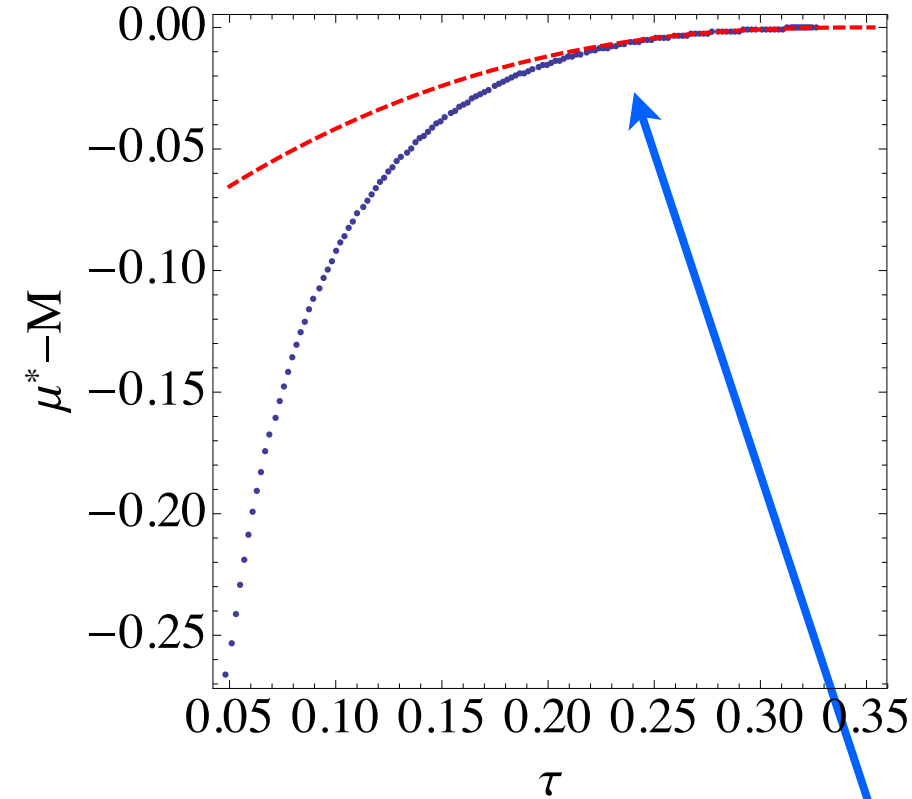


$$\epsilon = E(p) - M$$

Rapid growth of infrared occupation  
in a self-similar scaling fashion



# Self-Similar Scaling Analysis



Scaling from stationary cascade  
(c.f. Semikoz-Tkachev)

$$f(\varepsilon, \tau) = A^{-\alpha}(\tau) f_s(\varepsilon/A(\tau))$$

$$f(0, \tau) \propto [(\tau_c - \tau)(\alpha - 1)]^{-\alpha/2(\alpha-1)}$$

We have found consistent  
scaling exponents in this case.

Note: S-T uses classical limit of  
kinetic equations, while we  
maintain full quantum factors.

$$\mu^* = M - (\tau_c - \tau)^{\beta/(2\beta-2)}$$

$$\beta \simeq 1.25 \rightarrow \beta/(2\beta - 2) = 2.5$$

$$\tau_c \simeq 0.35$$

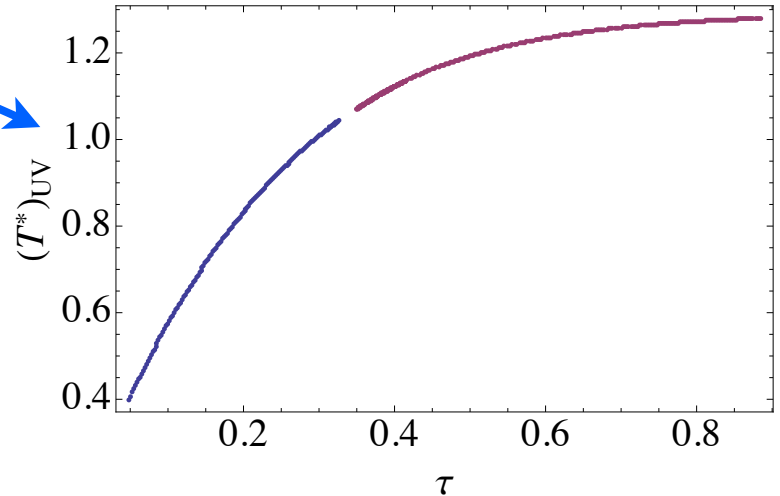
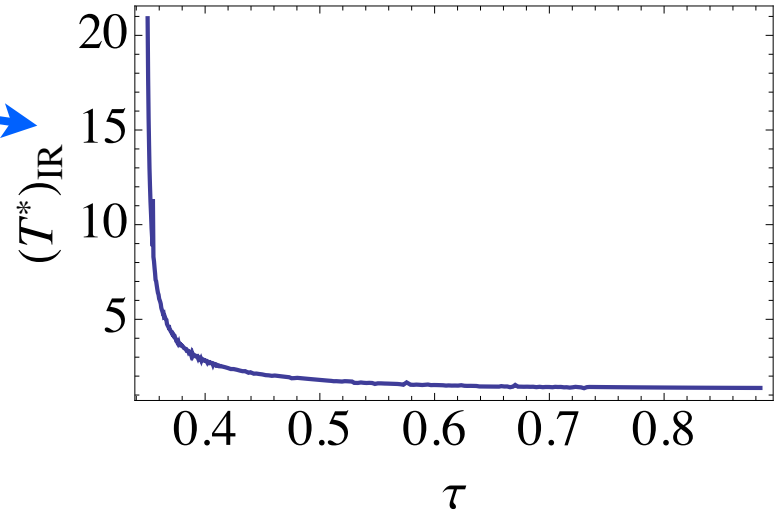
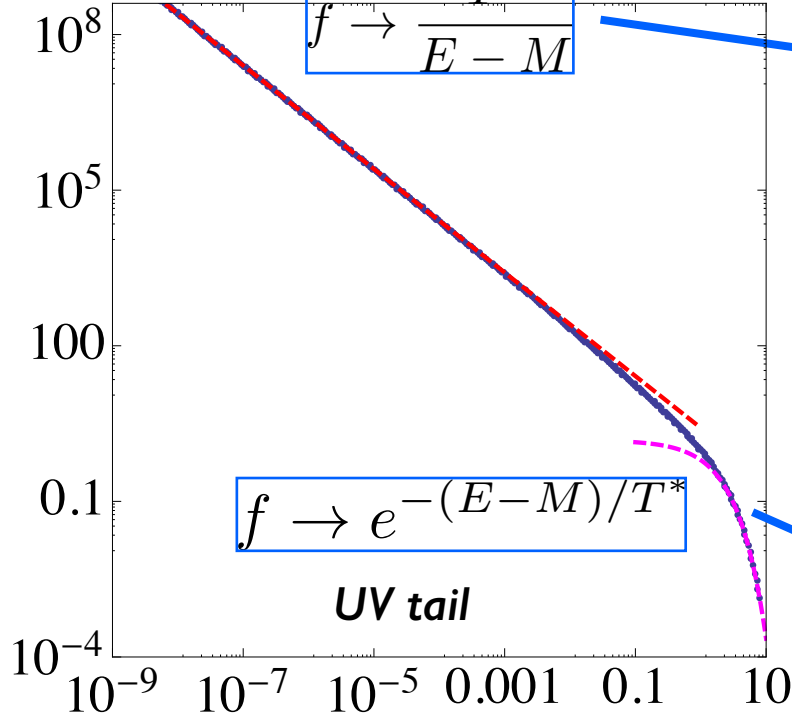
# Evolution after Onset of BEC

classical thermal @ IR

$$f \rightarrow \frac{T^*}{E - M}$$

$$f \rightarrow e^{-(E-M)/T^*}$$

UV tail



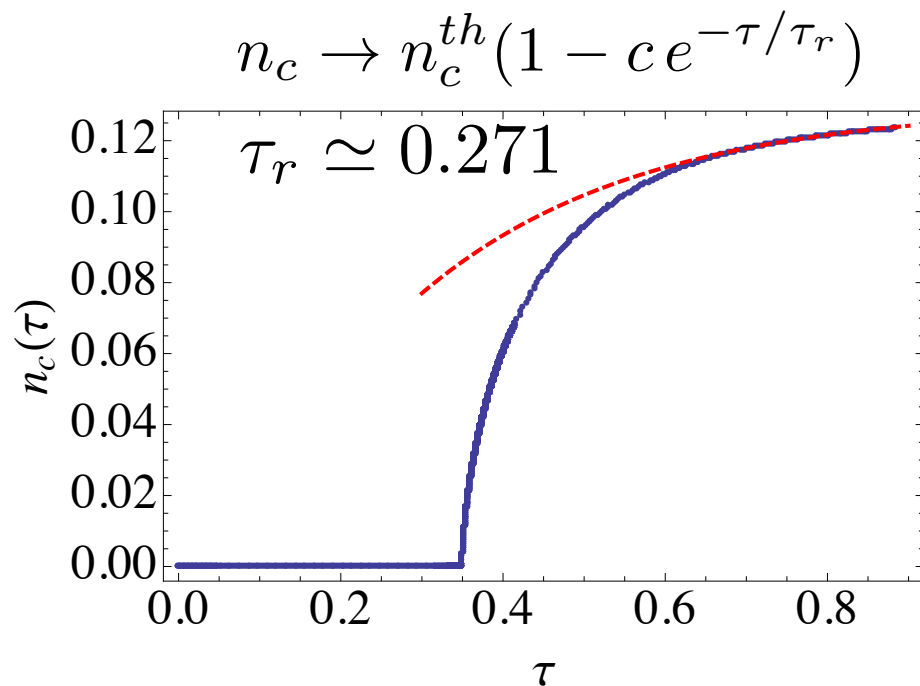
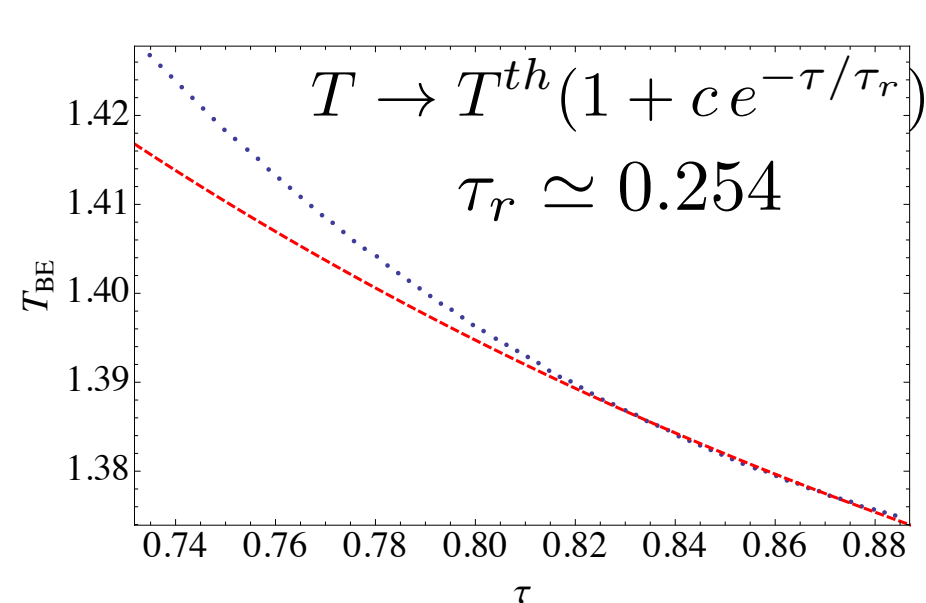
Two remarks:

\*  $f(p)$  gradually switches from  $f \sim 1/p^2$  toward  $f \sim 1/p$

\* McLerran parameterization

$$f \rightarrow \frac{\gamma}{e^{(E-M)/T} - 1}$$

# Final Approach toward Thermalization

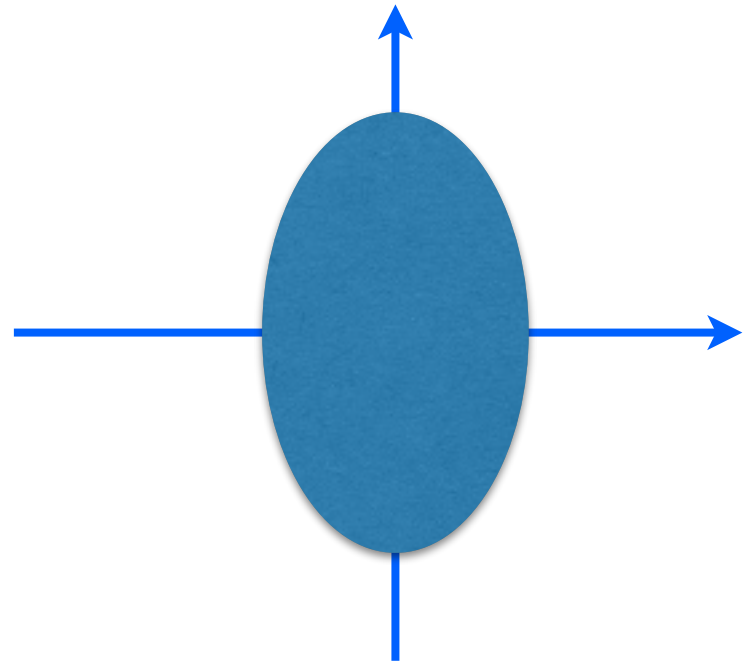
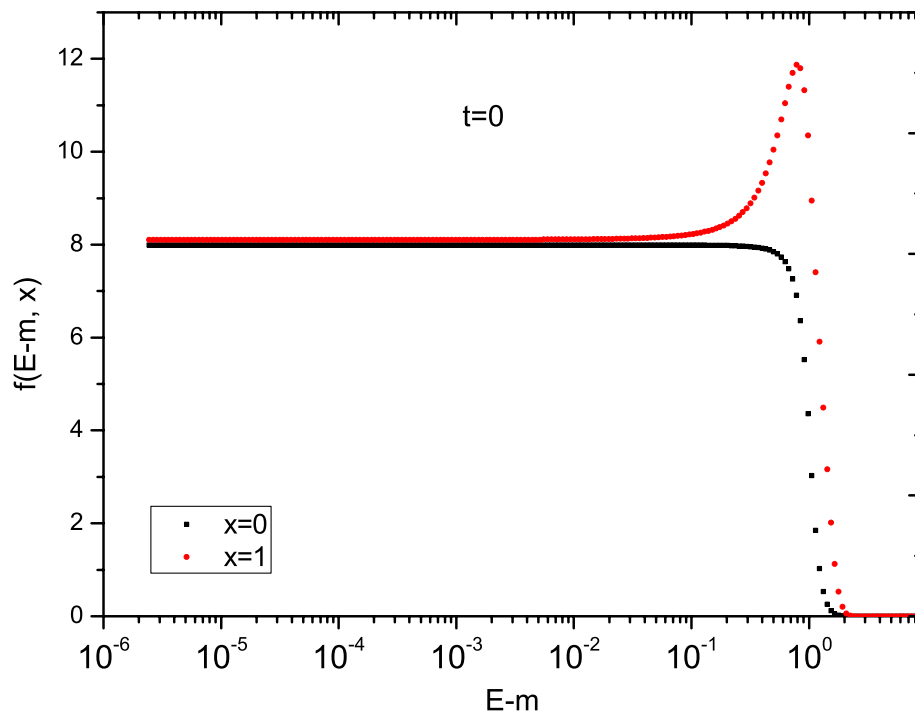


Pertinent time scale:

$$t = \tau \times \frac{64\pi^3}{\lambda^2} \quad 64\pi^3 \simeq 1984 \quad f_0 \sim 8$$

$$t_{th} \sim \hat{O}(10^{3 \sim 4})$$

# Anisotropic Initial Condition



**Interesting questions:**

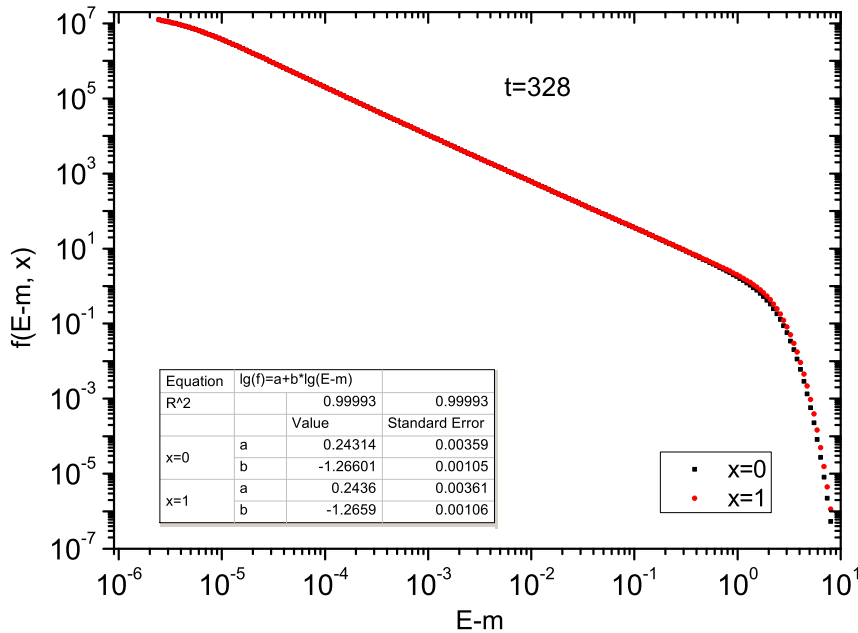
**\* How anisotropy affects evolution, particularly BEC onset?**

**\* How the system evolves toward isotropy?**

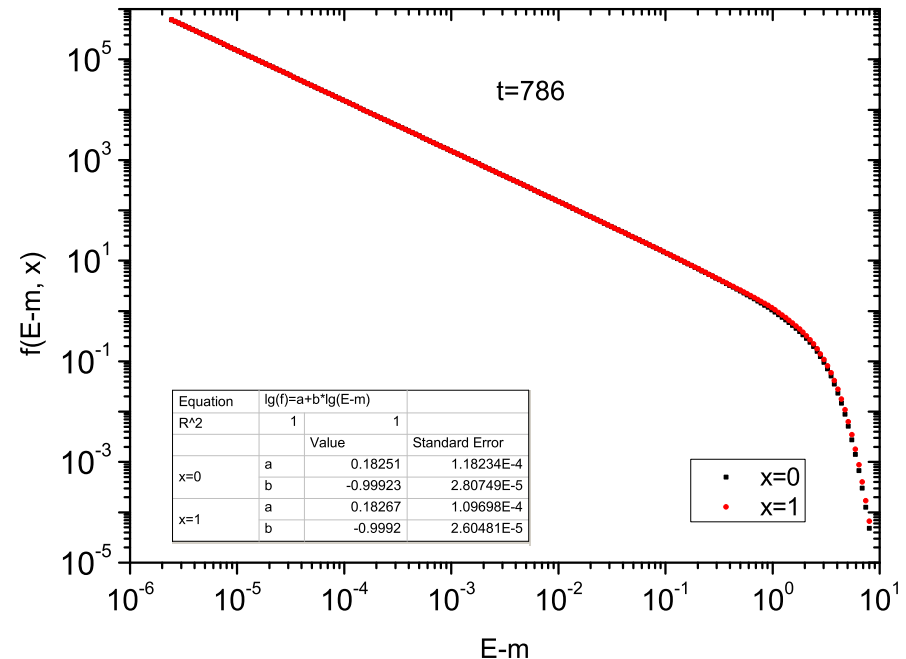
**[Note: static box for now, but anisotropic I.C.]**

# Evolution from Anisotropic I.C.

*just before onset time*

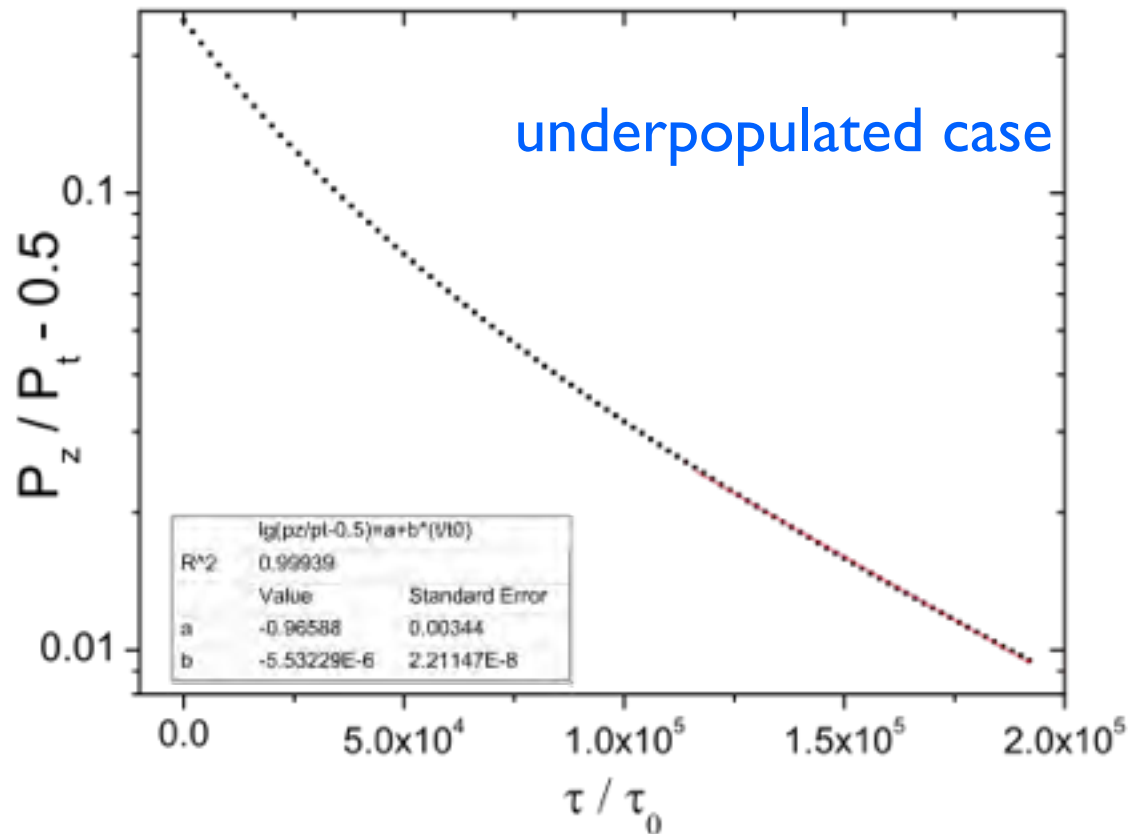


*very close to thermal point*



- \* IR part essentially maintains isotropy all the time
- \* Same IR self-similar scaling behavior before onset
- \* Same IR classical thermal after onset
- \* UV tails keep adjusting toward isotropy

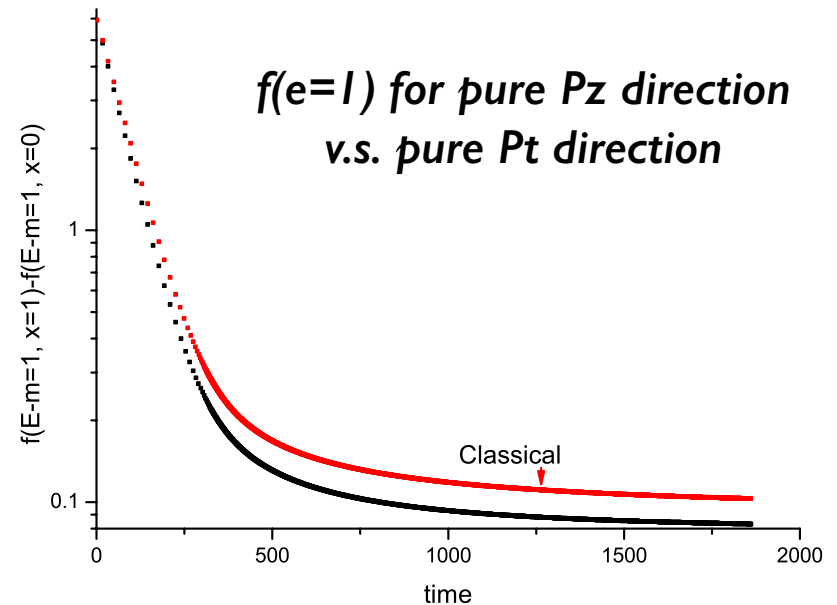
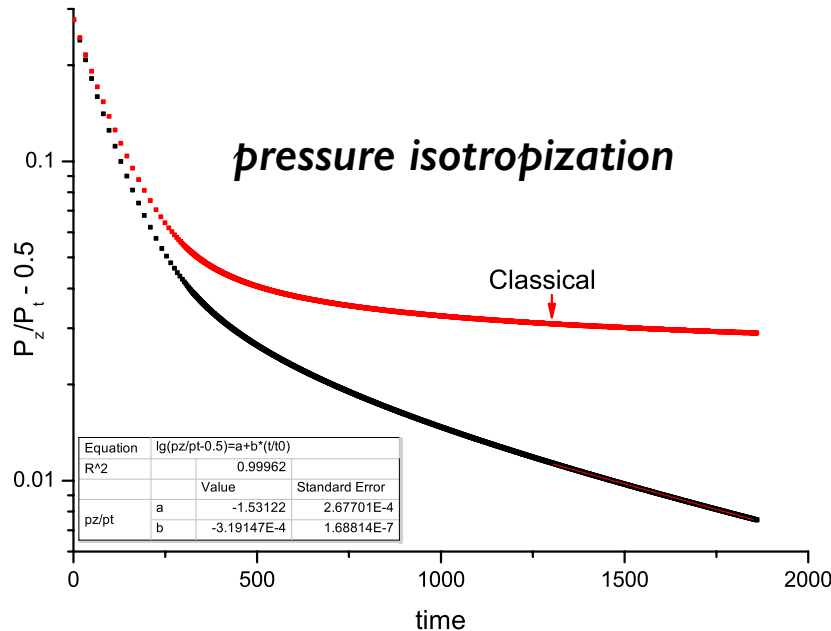
# Isotropization from Anisotropic I.C.



$$\tau_{iso} \sim \frac{64\pi^3}{\lambda^2 f_0^2}$$

# Isotropization: Classical v.s. Quantum

We now study the overpopulated case: in particular the comparison between the classical limit and the full quantum.



*The system appears to have difficulty with isotropization in the classical limit — WHY?*

*\* Isotropization mostly concerns  $\sim UV$  scale where occupation  $f \sim O(1)$  or even less*

*\* The classical approximation underestimates isotropizing scatterings:*

$$f_L f_L (1 + f_T) (1 + f_T) - f_T f_T (1 + f_L) (1 + f_L) = (2f_T f_L + f_L + f_T)(f_L - f_T)$$

# Summary

- \* Initial gluon system at very early stage of a heavy ion collision is characterized by **high overpopulation**.
- \* Elastic process (alone) in highly overpopulated system can induce **very rapid growth of soft modes** and drive toward equilibration. This is a very robust feature and may lead to a transient **Bose-Einstein Condensate**.
- \* Dynamical onset of BEC in a scaling way is found to be a very robust feature despite many details.
- \* Inelastic processes may further enhance the rapid growth of soft modes and **catalyze the onset** of BEC (but will remove the condensate afterwards). The time window for a condensate could be sizable.
- \* We hope to be able to include longitudinal expansion, and to quantitatively compare kinetic results with other approach soon.